1. Details of Module and its structure

Module Detail	
Subject Name	Physics
Course Name	Physics 01 (Physics Part-1, Class XI)
Module Name/Title	Unit 5, Module 8,Rolling Motion Chapter 7, System of particles and rotational motion
Module Id	Keph_10708_eContent
Pre-requisites	Torque, laws of motion, basic vector algebra
Objectives	After going through this lesson, the learners will be able to • **Understand** Rolling motion of a rigid body • **Visualize** the Constraints for pure rolling motion • **Deduce** Kinetic energy of rolling bodies • **Calculate** Acceleration of rolling bodies
Keywords	Translational motion, rotational motion, rolling motion, kinetic energy associated with rotational motion

2. Development Team

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1. UNIT SYLLABUS

Unit V: Motion of System of Particles and Rigid body

Chapter 7: System of particles and Rotational Motion

Centre of mass of a two-particle system; momentum conservation and centre of mass motion. Centre of mass of a rigid body; Centre of mass of a uniform rod.

Moment of a force; torque; angular momentum; Law of conservation of angular momentum and its applications.

Equilibrium of rigid bodies; rigid body rotation and equations of rotational motion; comparison of linear and rotational motions.

Moment of inertia; radius of gyration; values of moments of inertia for simple geometrical objects (no derivation). Statement of parallel and perpendicular axes theorems and their applications.

2. MODULE-WISE DISTRIBUTION OF UNIT SYLLABUS 8 Modules

The above unit is divided into eight modules for better understanding.

Module 1	• Rigid body • Centre of mass • Distribution of mass • Types of motion: Translatory, circulatory and rotatory
Module 2	• Centre of mass • Application of centre of mass to describe motion • Motion of centre of mass
Module 3	• Analogy of circular motion of a point particle about a point and different points on a rigid body about an axis • Relation v = rω • Kinematics of rotational motion
Module 4	• Moment of inertia • Difference between mass and moment of inertia • Derivation of value of moment of inertia for a lamina about a vertical axis perpendicular to the plane of the lamina • unit • Radius of gyration • Perpendicular and parallel axis theorems
Module 5	• Torque • Types of torque • Dynamics of rotator motion • $m = I\alpha$
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Module 8	• Rolling on plane surface • Rolling on Horizontal • Rolling on inclined surface • Applications

Module 8

3. WORDS YOU MUST KNOW

Let us remember the words we have been using in our study of this physics course.

- Rigid body: An object for which individual particles continue to be at the same separation over a period of time
- Point object: A point object is much smaller than the distance due to change in its position.
- Distance travelled: The distance an object has moved from its starting position. Its SI unit is m and it can be zero or positive.
- Displacement: The distance an object has moved from its starting position moves in a particular direction. Its SI unit is m and it can be zero, positive or negative.
- Speed: Rate of change of position and its unit m/s.
- Average speed: Average speed $=\frac{\text{Total path length}}{\text{Total time interval}}$

  Its unit is m/s.
- Velocity (v): Rate of change of position in a particular direction and its unit is m/s.
- Instantaneous velocity The velocity at an instant is defined as the limit of the average velocity as the time interval Δt becomes infinitesimally small velocity at any instant of time
- Uniform motion: object covers equal distance in equal interval of time
- Non uniform motion: object covers unequal distance in equal interval of time
- Acceleration (a): rate of change of velocity with time and its unit is m/s^2. Velocity may change due to change in its magnitude or change in its direction or change in both magnitude and direction.
- Constant acceleration: Acceleration which remains constant.
- Momentum (p): The impact capacity of a moving body is mv and its unit is kg m/s.
- Force (F): Something that changes the state of rest or uniform motion of a body. Unit of force is Newton. It is a vector, as it has magnitude which tells us the strength or magnitude of the force and the direction of force is very important

- Constant force: A force for which both magnitude and direction remain the same with passage of time
- Variable force: A force for which either magnitude or direction or both change with passage of time
- External unbalanced force: A single force or a resultant of many forces that act externally on an object.
- Dimensional formula: An expression which shows how and in which way the fundamental quantities like, mass (M), length (L) and time (T).
- Kinematics: Study of motion without involving the cause of motion
- Dynamics: Study of motion along with the cause producing the motion
- Vector: A physical quantity that has both magnitude and direction .displacement is a vector, force is a vector, acceleration is a vector etc.
- Vector algebra: Mathematical rules of adding, subtracting and multiplying vectors
- Resolution of vectors: A vector can be resolved in two mutually perpendicular directions. We used this for vector addition and in our study of motion in 2 and 3 dimensions.
- Dot product: two vectors on multiplication yield a scalar quantity. Dot product of vector A and B: A.B= $|A||B|cos\theta$ where θ is the angle between the two vectors. Dot product is a scalar quantity and has no direction. It can also be taken as the product of magnitude of A and the component of B along A or product of B and component of A along B.
- Work: Work is said to be done by an external force acting on a body if it produces displacement W= F.S cosθ, where work is the dot product of vector F(force) and Vector S (displacement) and θ is the angle between them . Its unit is joule and dimensional formula is ML^2T^{-2}. It can also be stated as the product of component of the force in the direction of displacement and the magnitude of displacement. Work can be done by constant or variable force and work can be zero, positive or negative.
- Energy: The ability of a body to do work. Heat, light, chemical, nuclear, mechanical are different types of energy. Energy can never be created or destroyed it only changes from one form to the other.

- Kinetic Energy: The energy possessed by a body due to its motion = ½ mv^2, where 'm' is the mass of the body and 'v' is the velocity of the body at the instant its kinetic energy is being calculated.
- Conservative force: A force is said to be conservative if the work done by the force in displacing a body from one point to another is independent of the path followed by the particle and depends on the end points. Example: gravitational force.
- Non- conservative forces: **A** force is said to be non-conservative if: the work done by it on an object depends on the path and the work done by it through any round trip is not zero. Example: friction.
- Work Energy theorem: Relates work done on a body to the change in mechanical energy of a body i.e., $W = F.S = \frac{1}{2} mV_f^2 - \frac{1}{2} mV_i^2$
- Conservation of mechanical energy: Mechanical energy is conserved if work done is by conservative forces.
- Potential energy due to position: Work done in raising the object of mass m to a particular height (distance less than radius of the earth) = 'mgh'.
- Collision: Sudden interaction between two or more objects. We are only considering two body collisions.
- Collision in one dimension: Interacting bodies move along the same straight path before and after collision.
- Elastic collision: Collision in which both momentum and kinetic energy is conserved.
- Inelastic collision: Momentum is conserved but kinetic energy is not conserved.
- Coefficient of restitution: The ratio of relative velocity after the collision and relative velocity before collision. Its value ranges from 0-1.
- Torque: It is rotational analogous of Force and it has following characteristics:

  i. Torque is the turning effect of the force about the given axis of rotation.
  ii. The torque equals the moment of the force about the given axis/ point of rotation.

iii. Just as force equals the rate of change of linear momentum, torque equals the rate of change of angular momentum.

- Mechanical equilibrium: It implies either the object is at rest and stays at rest is said to be in static mechanical equilibrium or the center of mass, of the system, moves with a constant velocity is said to be in dynamic mechanical equilibrium.
- First condition for equilibrium: is that the center of mass of the body has zero acceleration; this happens when if the vector sum of all external forces acting on the body is zero. It is also called the condition for the translatory equilibrium. In vector and component forms, we can write:

$$\overrightarrow{F_{net}} = 0$$

- Second condition for an extended body to be in equilibrium is that the body must have no tendency to rotate. It is also called the condition for the rotational equilibrium. In vector and component forms, we can write:

$$\overrightarrow{\tau_{net}} = 0$$

- Center of gravity: It is that point in a given body, around which the resultant torque, due to the gravity forces, vanishes. The concept can be useful in designing structures, especially buildings, bridges etc., so that they remain stable under the influence of the forces acting on them.
- Angular momentum: The angular momentum, of a system of n particles about the origin, is given by:

$$\overrightarrow{\ell_{net}} = \sum_{i=1}^{n} \overrightarrow{r_i} \text{ X } \overrightarrow{p_i}$$

Here, $\overrightarrow{r_i}$ is the position vector of ith particle whose linear momentum is given by $\overrightarrow{p_i}$

For a rigid body the angular momentum, about an axis can be expressed as:

$$\overrightarrow{\ell_{net}} = I\vec{\omega}$$

4. INTRODUCTION

Watching a moving wheel may make us think. **Watching a moving wheel may make you think deeper**!!

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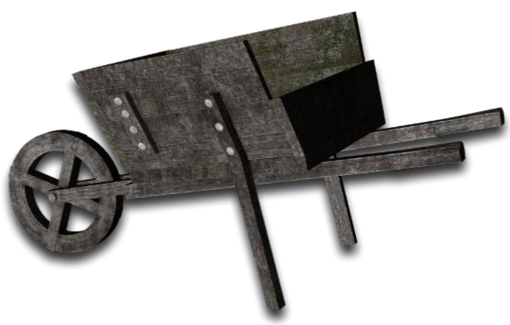


https://www.needpix.com/photo/891097/wheelbarrow-rowern-wood-cart-wooden-wheels-transport

The wheel, as we all know, is a wonderful invention that has helped and revolutionized transportation over the centuries. In fact, the motion of a wheel, one of the most common motions, observed in daily life. If watched carefully the wheel moves overall along a (nearly) straight path, most of the times, however it is simultaneously rotating also.

5. ROLLING MOTION AS A SUPERPOSITION OF TRANSLATIONAL AND ROTATIONAL MOTION

Where shall we place it in our categories of motion translational, or rotational, or both? Do the mathematical results, we have derived, for linear motion and circular motion, can be applied directly here, or they need some modification?

When an object shows both translational and rotational motion simultaneously, it is said to have a rolling motion.

https://thumbs.dreamstime.com/z/hands-rolling-pin-roll-dough-pizza-teenage-boy-53341098.jpg

watch a roti being rolled at home , observe the motion of the hands , rolling pin and the dough.

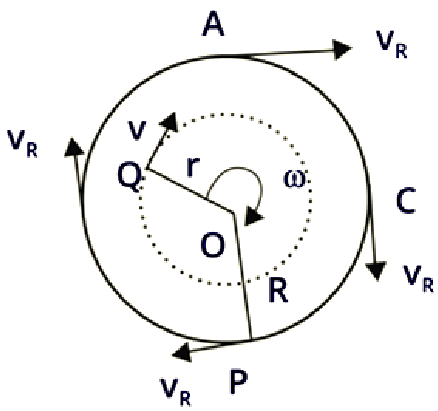
We shall begin with the case of a disc rolling along a straight path. The result however, will apply to any rolling body like circular or spherical bodies such as sphere, cylinder, ring etc. rolling on a level surface.

The rolling motion can be viewed as a superposition of two types of motion, namely translational and rotational.

It therefore needs visualization from both these perspectives. If one carefully watches a wheel rolling on a flat surface, its motion is a combination of pure rotation (i.e. rotation about a axis passing through its centre and parallel to ground) and pure translation (i.e. the centre of mass of the wheel shows a translation motion along a straight path).Can you observe the two motions in the ‘belan’

Let us first consider the rotational aspect of a rolling disc or wheel. Consider a disc having a pure rotation, about a fixed axis passing through its centre; let its angular velocity be "ω". Each particle of the rotating (rigid) disc, moves with the same angular velocity. In the figure, we have shown the (corresponding) linear velocities of particles, occupying four positions on the rim, with appropriate vectors, due to this rotational motion of the disc. The magnitude of velocity of these four particles A, B, C and P on the rim, resulting from pure rotation, is given by:

$$\mathbf{v_{rotational}} = \mathbf{v_R} = \boldsymbol{\omega}\mathbf{R} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$



Pure Rotation

However, if we consider a particle Q, inside the disc, at a distance "r" from the centre O, the magnitude of its linear velocity, resulting from pure rotation is given by :

$$\mathbf{v} = \boldsymbol{\omega}\mathbf{r} \ \ldots\ldots\ldots\ldots\ldots(2)$$

The direction of linear velocity due to the rotational motion of the disc, is along the tangent to the circular path traced by the particle under consideration as shown in the figure above.

Substituting the value of ω, from equation (1) in equation (2), we can obtain the velocity of a particle inside the rotating disc as:

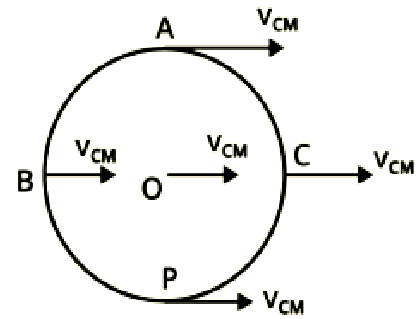
$$\mathbf{v} = \frac{\mathbf{v_R}}{\mathbf{R}}\mathbf{r}$$

where "r" is the linear distance of the position occupied by the (inner) particle from the axis of rotation and "R" is the radius of the disc. Therefore the linear velocities of different particles due to the rotational motion of disc, are different, however, they all have the same angular velocity (ω).

Now consider the given disc to have a pure translatory motion. Each particle, of the disc now translating with a linear velocity equal to the linear velocity of the center of mass, " v_{CM} ".

$$\mathbf{v}_{\mathbf{translation}} = \mathbf{v}_{\mathbf{CM}}$$

Unlike the case of pure rotation, each of the particle-whether present on the rim, or within the disc – would now be moving with the same linear velocity. In the figure given below, the linear velocities using appropriate vectors of different particles, are shown for four particles A, B, C and P, all on the rim of the disc.



Pure Translation

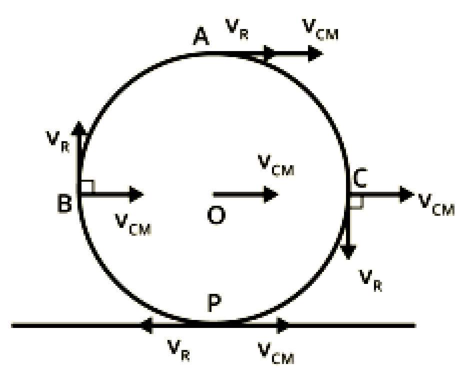
Let us now consider a disc whose translation motion has been superposed on its rotational motion, discussed above. The net velocity of the particles on the disc will now be obtained by adding the two velocity vectors; one due to pure rotation and the other due to pure translation.

i.e. net velocity of any particle on disc**:**

$$\overrightarrow{\mathbf{v}_{\mathbf{net}}} = \overrightarrow{\mathbf{v}_{\mathbf{R}}} + \overrightarrow{\mathbf{v}_{\mathbf{CM}}}$$

NOTE

- **The bottom most point P, which is also the point of contact (with the ground), has a forward velocity $\overrightarrow{\mathbf{v}_{\mathbf{CM}}}$, because of translation and a velocity $\overrightarrow{\mathbf{v}_{\mathbf{R}}}$ (backwards) due to rotation.**
- **The two velocities at point B and C are perpendicular to each other;**
- **At point A, the top most point the two velocities are along the same direction, as shown in the figure**
- **Depending upon the relative magnitudes of $\overrightarrow{\mathbf{v}_{\mathbf{R}}}$ and $\overrightarrow{\mathbf{v}_{\mathbf{CM}}}$ we can have different type of rolling motion.**



(a) **When $|\vec{v}_R| = R\omega = |\vec{v}_{CM}|$ i.e., if the translational velocity, V_{CM} of the centre of mass has a magnitude R times the angular velocity ω, we say that the rolling object is in a state of pure rolling.**

It would mean that the velocity of the bottom most point or the point of contact (P), will be zero.

This is how we define pure rolling.

It also means that the frictional force will either not present at the point of contact, or only the static frictional force will be present.

In pure rolling, there is no relative motion between the two surfaces, at the point of contact.

(b) $|\vec{v}_R| = R\omega < |\vec{v}_{CM}|$, it means that the disc is translating faster as compared to its rotational motion. In such a case, for the point of contact we have a non zero velocity, in the forward direction. This is referred to as **forward slipping**.

For example, on a wet road, when the cycle tyres slide more but rotate less, the cycle can have a forward slipping.

(c) $|\vec{v}_R| = R\omega > |\vec{v}_{CM}|$. It means that the disc is translating slower as compared to its rotational motion. In such a case for the point of contact we have a non zero velocity, in the backward direction. This is referred to as **backward slipping**.

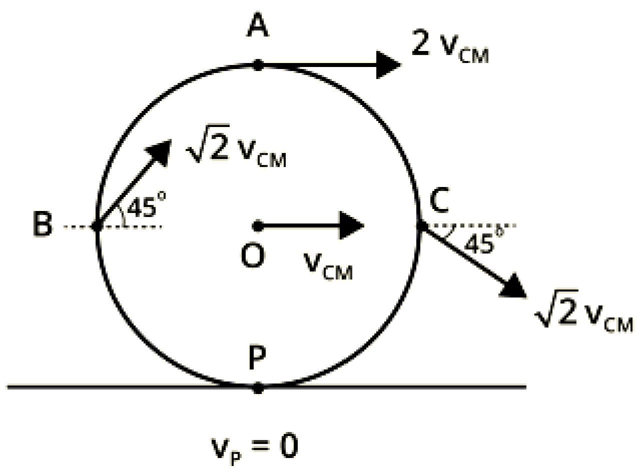
For example, on a muddy road, when the car tyres rotate more, but move (or slide) less, the car is said to have a backward slipping.

6. PURE ROLLING MOTION AND ITS CONSTRAINTS

We note here, that the particle, at the point of contact has zero linear velocity when an object has a pure rolling motion.

It is to be noted here that the point of contact has **zero 'instantaneous velocity'** resulting from its equal and opposite linear velocities, due to the pure rotation and pure translation parts of its motion. Nevertheless, it has finite angular velocity, "ω ", it is this that changes its position with the time. The moment the particle occupying the contact position changes its position, it acquires a finite linear velocity this is because the particle is no more at the contact point and the velocities resulting from its two constituent motions, are no longer equal and opposite.

The velocities of four particles A, B, C and P, all taken at the rim of a disc, having a pure rolling motion, are shown below.



For particle P, at point contact

$$\mathbf{v_{net,P}} = \mathbf{o}$$

For particle A, At topmost point

$$\mathbf{v_{net,A}} = \mathbf{2\ v_{CM}} \quad \text{(As } \overrightarrow{\mathbf{v_R}} \textbf{ and } \overrightarrow{\mathbf{v_{CM}}} \text{ are in same direction.)}$$

For particles B and C

$$\mathbf{v_{net,B}} = \sqrt{\mathbf{2}}\ \mathbf{v_{CM}} \quad \text{(As } \overrightarrow{\mathbf{v_R}} \textbf{ and } \overrightarrow{\mathbf{v_{CM}}} \text{ are perpendicular to each other.)}$$

Therefore, the net velocity, of a particle on the disc, depends on its position on the rim of the disc.

Let us now recall the constraints of rolling motion; these are as follows:

(i) The translational speed, of the centre of mass of a rolling object is related with its angular speed, about its centre, as

$$|\overrightarrow{\mathbf{v}_{\mathbf{CM}}}| = \boldsymbol{R\omega}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

(ii) If equation (3) is differentiated with respect to time one gets

$$|\overrightarrow{\mathbf{a}_{\mathbf{CM}}}| = \boldsymbol{R\alpha}\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(4)$$

The magnitude of the acceleration of the centre of mass is equal to the product of the radius "R" of the disc and the magnitude of the angular acceleration "α" , of the object about its centre of mass.

(iii)If one integrates equation (3) with respect to time one gets:

$$\left|\overrightarrow{\mathbf{\Delta X}_{\mathbf{CM}}}\right| = \boldsymbol{R\,\Delta\theta}$$

The magnitude of the linear displacement of the centre of mass, is equal to the product of the radius (R) and the angular displacement ($\Delta\theta$) of the object about the axis through its centre of mass.

7. INSTANTANEOUS CENTRE OF A ROLLING BODY

For a disc, undergoing pure rolling motion (as discussed above), we know that the instantaneous point of contact P of the disc with the surface does not slip. This means that the point P has zero velocity with respect to point O. Thus, at the instant the point P on the disc comes in contact with the surface it becomes the instantaneous centre.

The net speed of point A, the (instantaneous) topmost point is given by:

$$\mathbf{v}_{\mathbf{net,A}} = \mathbf{2\ v}_{\mathbf{CM}} = \mathbf{2\ R\ \omega} = \mathbf{AP}\ \times\ \boldsymbol{\omega}$$

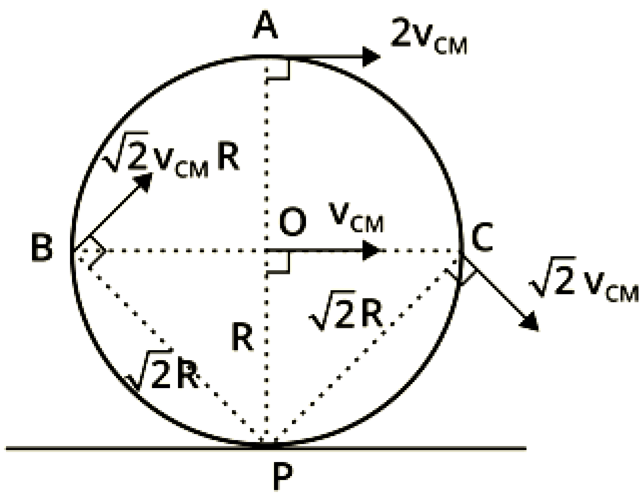
Also the velocity vector of the point A, the topmost point on the disc is perpendicular to the segment AP.

Similarly, for points B and C,

$$\mathbf{v_{net,B}} = \sqrt{2}\,\mathbf{v_{CM}} = \sqrt{2}\,\mathbf{R\omega} = \mathbf{BP} \times \boldsymbol{\omega}$$

Also one can see that the velocities, at these location, are perpendicular to the segment joining the point of contact P with the point B or C on the disc.

This shows that the point of contact acts as the instantaneous centre for a rolling disc i.e. the disc can be regarded as exhibiting pure rotation about the (instantaneous) point of contact P. For a person, standing on ground the disc is rolling. This fact can be used to determine velocities of other points on the disc. The velocity vectors for a number of points are given in figure below.



An interesting thing to note here is this: The more distant a point, on the disc is from the instantaneous centre P, the larger is the magnitude of its net speed:

$$\mathbf{v_{net}} = \boldsymbol{\omega}\,\mathbf{d}.$$

Here, "ω" is the angular velocity of the object and "d" is the distance of the point on the disc, from the point of contact.

THINK ABOUT THIS

A disc rotating about its axis with an angular speed ω_o, is placed lightly (without any translational push) on a perfectly frictionless table. The radius of the disc is R.

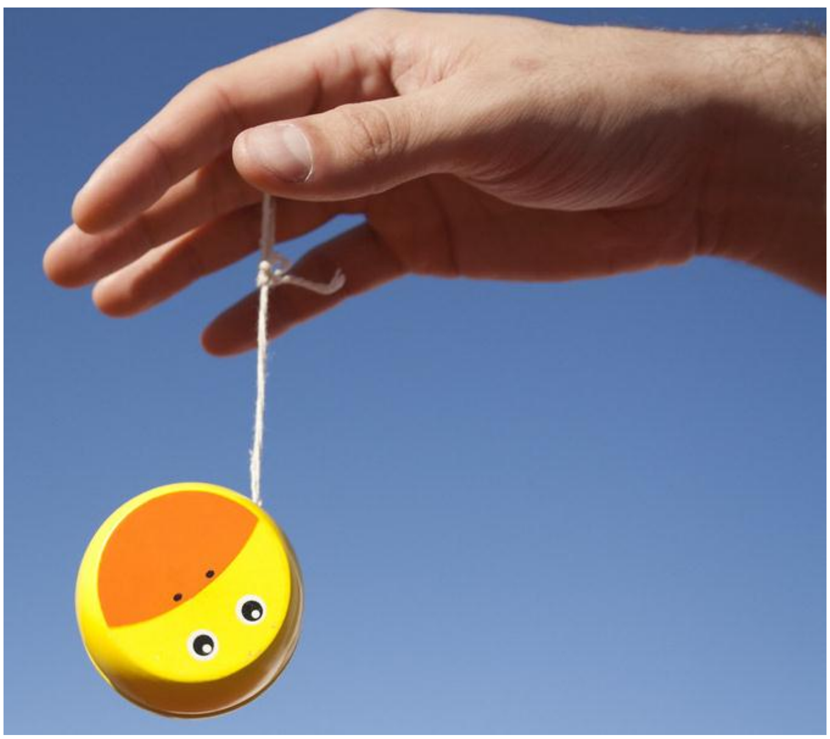
(a) **What are the linear velocities of the points A, B and C on the disc as shown in the figure?**

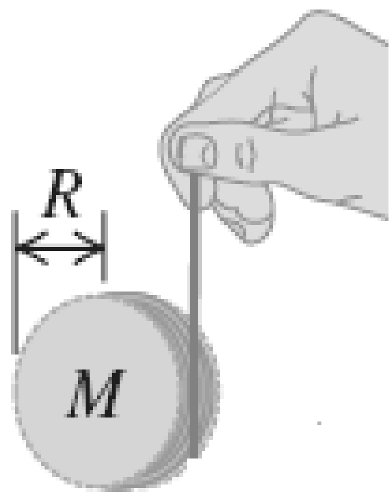
(b) **Will the disc roll in the direction indicated?**

Let's do some problems to apply the concepts studied above.

EXAMPLE:

A solid disc of mass M has a thin (light and inextensible) string wrapped several times around its circumference. The string is fixed at one end and the disc is released so that it falls vertically downwards. Determine the magnitude of the downward acceleration of the disc and tension in the string.





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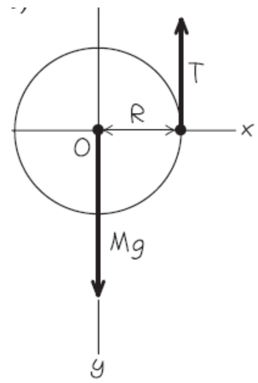
SOLUTION:

The forces acting on the disc are shown below. There are two forces acting on the disc namely

(i) Weight Mg, acting downwards

(ii) Tension T, acting upwards



The moment of inertia of the disc about its centre of mass, O is $\mathbf{I_{CM} = \frac{1}{2} M R^2}$

For translational motion of the disc, we will apply

$$\overrightarrow{\mathbf{F_{net}}} = \boldsymbol{M}\,\vec{\mathbf{g}} + \vec{\mathbf{T}}$$

$\therefore \overrightarrow{\mathbf{F_{net}}} = -\,\mathbf{Mg}\,\hat{\mathbf{j}} + \mathbf{T}\,\hat{\mathbf{j}}$

or $\mathbf{F_{net}} = \mathbf{Mg} - \mathbf{T} = \mathbf{M\,a_{CM}}$ ……………………..(1)

Here, $\mathbf{a_{CM}}$ is the acceleration of the centre of mass of the disc.

For the rotational motion of the disc about its centre of mass O we write

$$\overrightarrow{\boldsymbol{\tau}_{\mathbf{net}}} = \overrightarrow{\boldsymbol{\tau}_{\mathbf{1}}} + \overrightarrow{\boldsymbol{\tau}_{\mathbf{2}}}$$

Here, $\overrightarrow{\boldsymbol{\tau}_{\mathbf{1}}}$ is the torque due to Mg, this is zero as the force passes through the axis of rotation.

Also, $\overrightarrow{\boldsymbol{\tau}_{\mathbf{2}}}$ is the torque due to tension T; its magnitude is given as:

$|\overrightarrow{\boldsymbol{\tau}_{\mathbf{2}}}| = \mathbf{R\,T\,sin90^o} = \mathbf{R\,T}$, in clockwise direction

Therefore, the magnitude of the net torque can be written as:

$|\overrightarrow{\boldsymbol{\tau}_{\mathbf{net}}}| = \mathbf{R\,T} = \mathbf{I_{CM}}\,\boldsymbol{\alpha}$ …………….(2)

Here, α is the angular acceleration of the disc.

Using equation (1) and (2) one can write

$\mathbf{M\,a_{CM} = Mg - \frac{I_{CM}\,\alpha}{R}}$……….(3)

Also, for a pure rolling motion $|\overrightarrow{a_{CM}}| = \mathbf{R\alpha}$

Substituting the value of $\boldsymbol{\alpha} = \frac{a_{CM}}{R}$ in equation (3) we get

$$\mathbf{a_{CM} = \frac{M\,R^2}{M\,R^2 + I_{CM}}\,g}$$

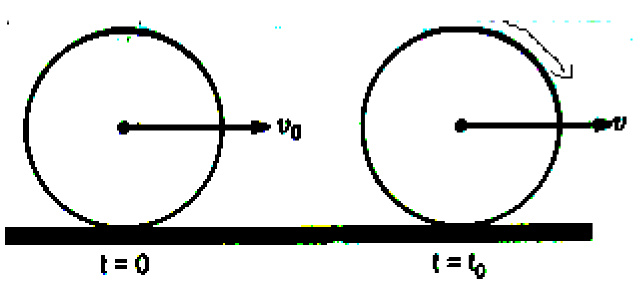
Substitute $\mathbf{I_{CM} = \frac{1}{2}M\,R^2}$ we get $\mathbf{a_{CM} = 2g/3}$

For finding the value of tension T, use equation (3)

$\mathbf{T = \frac{I_{CM}\,\alpha}{R} = \frac{\frac{1}{2}M\,R^2 \times a_{CM}}{R \times R}}$ = Mg/3 $\quad (since, \boldsymbol{\alpha} = \frac{a_{CM}}{R})$

EXAMPLE:

A uniform ring of mass M and radius R is projected horizontally with a velocity v_o ,on a rough horizontal floor. It starts off with a purely sliding motion at t=0. At t= t_o, the ring shows a pure rolling motion, as shown in the figure.



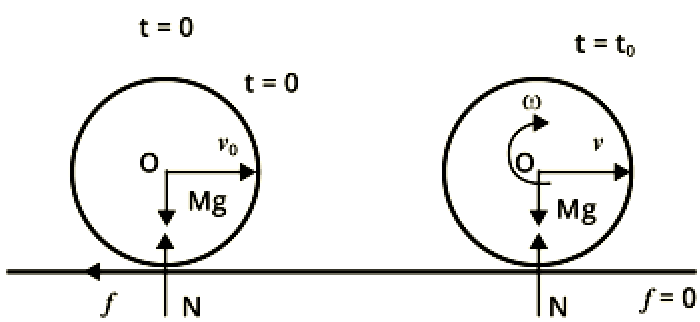
(i) Calculate the velocity of the centre of mass of the disc at t= t_o

(ii) Assuming the coefficient of kinetic friction to be μ, calculate the time t_o taken by the disc having a pure rolling motion.

SOLUTION:

During the time interval t=0 to t= t_o, there is forward sliding as the translational motion dominates over the rotational motion. The frictional force is kinetic in nature and its direction will be opposite to the velocity of the centre of mass. For this interval, the different forces acting on the body are:

(a) Weight Mg (vertically, down)
(b) Normal force N (vertically, up)
(c) Kinetic friction force, f (horizontally, left)



The net force on the body, determines the acceleration of centre of mass.

Along y axis: $\mathbf{F_{net,y} = 0 = N - Mg}$

N= Mg

Along x axis: $\mathbf{F_{net,x}} = -f = \mathbf{M\,a_{CM}}$

Also $\mathbf{f = \mu N = \mu M g}$

Therefore, the acceleration of the centre of mass, is:

$$a_{CM} = \frac{-f}{M} = \frac{-\mu N}{M} = \frac{-\mu M g}{M} = -\mu g$$

If v is the velocity of centre of mass, at (t= t_o), we can write:

$$v = u + a\,t = v_o - \mu\, g\, t_0 \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

The net torque on the body will help us to find the angular acceleration of the body during the interval t= 0 to t= t_o.

There are three forces acting on the body, as described above. The torque on the body due to them is given respectively, as:

(a) Torque due to weight Mg will be zero as it passes through the centre of mass of the body.
(b) Torque due to normal reaction N of the floor, will also be zero as it also passes through the centre of mass of the body.
(c) Torque due to friction can be written as

$$\tau = r\,F\,\sin\theta = R\,f\,\sin 90^o$$

$$\tau = R\,\mu\,M\,g$$

Now, $\tau = I\,\alpha = \text{moment of inertia} \times \text{angular acceleration}$

$$\alpha = \frac{\tau}{I} = \frac{R\,\mu\,M\,g}{I}$$

Let ω be the angular velocity of the body at t= t_o. Using the equation of motion, we can write

$$\omega = \omega_o + \alpha\, t = 0 + \frac{R\,\mu\,M\,g}{I}\, t_0 \ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

The motion of the body becomes a pure rolling motion at t= t_o, hence, using the constraint of pure rolling, we can write:

$$|\overrightarrow{v_{CM}}| = R\omega$$

From equation (1) and (2) using the above constraint, we get:

$$v_o - \mu\, g\, t_0 = R \times \frac{R\,\mu\,M\,g}{I}\, t_0$$

Therefore,

$$\mathbf{t_0} = \frac{\mathbf{v_o}}{\mu\mathbf{g}\left(\mathbf{1} + \frac{\mathbf{M\,R^2}}{\mathbf{I}}\right)}$$

Now, $\frac{\mathbf{I}}{\mathbf{M\,R^2}} = \mathbf{k^2}/R^2$, where K is radius of gyration

Time taken to attain pure rolling $\mathbf{t_0} = \frac{\mathbf{v_o}}{\mu\mathbf{g}(\mathbf{1}+\frac{\mathbf{R^2}}{\mathbf{k^2}})}$.............................. (3)

Using equation (1) and (3) speed of centre of mass of the rolling body can be determined at $t = t_o$

$$v = v_o - \mu\, g\, \frac{\mathbf{v_o}}{\mu\mathbf{g}(\mathbf{1}+\frac{\mathbf{R^2}}{\mathbf{k^2}})} = \frac{\mathbf{v_o K^2}}{\mathbf{k^2+R^2}} \quad \text{...............} \qquad (4)$$

Expression (3) and (4) can be used as general equations.

Here the rolling object is ring, therefore

$$\frac{\mathbf{I}}{\mathbf{M\,R^2}} = \mathrm{M}\mathbf{R^2} = \frac{\mathbf{M\,K^2}}{\mathbf{M\,R^2}} = \mathbf{1\ and\ K = R}$$

Hence, the speed of centre of mass, of the ring, at $t = t_o$ is:

$$\mathbf{v} = \frac{\mathbf{v_o}\mathrm{K}^2}{\mathbf{k^2+1}} = \mathbf{v_o/2}$$

Also, Time taken to attain pure rolling by the ring is:

$$\mathbf{t_0} = \frac{\mathbf{v_o}}{\mu\mathbf{g}(\mathbf{1}+\frac{\mathbf{R^2}}{\mathbf{k^2}})} = \frac{\mathbf{v_o}}{\mathbf{2}\,\mu\mathbf{g}}$$

8. KINETIC ENERGY OF A ROLLING BODY

As we have seen the motion of a rolling body can be considered as a superposition of a translational and a rotational motion. For a body of mass M, let the centre of mass be moving with a velocity of v_{CM} .Also, let it be is rotating simultaneously, about its centre of mass, with an angular velocity ω. The kinetic energy of this rolling body can then be written as:

$$\mathbf{KE_{Rolling}} = \mathbf{KE_{translational}} + \mathbf{KE_{rotaional}}$$

$$\mathbf{KE_{Rolling}} = \frac{1}{2}\,\mathbf{M\,v_{CM}^2} + \frac{1}{2}\,\mathbf{I_{CM}}\,\omega^2 \ldots\ldots\ldots\ldots \qquad (1)$$

Here, $\mathbf{I_{CM}}$ is the moment of inertia of the body about its (rotational) axis, pasing through its centre of mass.

For a body, having a pure rolling motion, the speed of centre of mass is related with its angular speed as:

$$\mathbf{v_{CM}} = \mathbf{R\,\omega} \ldots\ldots\ldots\ldots \qquad (2)$$

From equation (1) and (2) kinetic energy of rolling body can be written as

$$\mathbf{KE_{Rolling}} = \frac{1}{2}\,\mathbf{M\,(R\,\omega)^2} + \frac{1}{2}\,\mathbf{I_{CM}}\,\omega^2$$

$$\mathbf{KE_{Rolling}} = \frac{1}{2}\,\left[\mathbf{M\,R^2} + \mathbf{I_{CM}}\right]\boldsymbol{\omega}^2 \ldots\ldots\ldots\ldots\ldots\ldots \qquad (3)$$

Using parallel axis theorem, the moment of inertia of the body about the (instantaneous) point of contact, is given by:

$$\mathbf{I_P} = \mathbf{M\,R^2} + \mathbf{I_{CM}} \ldots\ldots\ldots\ldots\ldots\ldots \qquad (4)$$

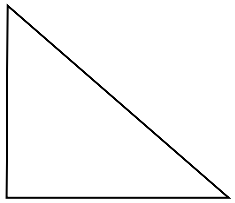
From equation (3) and (4) we can write

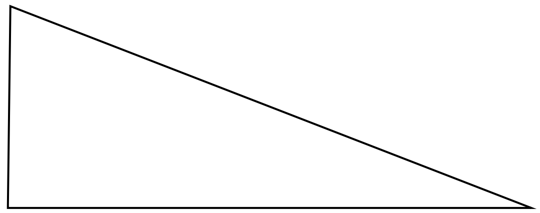
$$\mathbf{KE_{Rolling}} = \frac{1}{2}\,\mathbf{I_P}\,\boldsymbol{\omega}^2 \ldots\ldots\ldots\ldots\ldots \qquad (5)$$

Hence one can express the kinetic energy of a rolling body either as **sum of its translational kinetic energy and its rotational kinetic energy as given by the equation (1) or as just equal to its rotational kinetic energy, about the point of contact as given by equation (5).**

THINK ABOUT THIS

A solid sphere rolls down two different inclined planes, of the same height but having different angles of inclination.





(a) Will it reach the bottom with the same speed in each case?

(b) Will it take longer to roll down one plane than the other?

(c) If so, which one and why?

EXAMPLE:

A ball of radius R and mass M is rolling without slipping on a horizontal surface with a speed v. It then rolls without slipping up an inclined plane to a height h before momentarily coming to rest. Determine the value of h.

SOLUTION:

Here, the frictional force involved is the force of static friction during pure rolling, the instantaneous point of contact, is (momentarily) at rest. The concept of conservation of mechanical energy can be therefore, used here.

$$\textbf{Mechanical Energy}_{\textbf{initial}} = \textbf{Mechanical Energy}_{\textbf{final}}$$

$$\mathbf{KE_i + U_i = KE_f + U_f}$$

Considering the horizontal level of centre of mass to be the reference level of potential energy, we have:

$$\mathbf{KE_i = \frac{1}{2}\, M\, v_{CM}^2 + \frac{1}{2}\, I_{CM}\, \omega^2}$$

$$\mathbf{U_i = 0}$$

$$\textbf{Also},\qquad \mathbf{KE_f = 0}$$

$$\mathbf{U_f = Mgh}$$

Therefore

$$\mathbf{\left(\frac{1}{2}\, M\, v_{CM}^2 + \frac{1}{2}\, I_{CM}\, \omega^2\right) + 0 = 0 + Mgh}$$

Substitute $\boldsymbol{\omega} = \frac{\mathbf{V_{CM}}}{\mathbf{R}}$, the motion being a pure rolling motion, we get:

$$h = \frac{\frac{1}{2}\left(M\,v_{CM}^2 + I_{CM}\,\frac{v_{CM}^2}{R^2}\right)}{M\,g}$$

$$h = \frac{v_{CM}^2\left(1 + \frac{I_{CM}}{M\,R^2}\right)}{2\,g} \ldots\ldots\ldots\ldots\ldots$$

The above expression can be used as general expression for any rolling object. One can conclude that a body of higher moment of inertia, about its centre of mass axis for a given value of mass and radius will rise up to a greater height. The factor $\frac{I_{CM}}{M\,R^2}$ is also equal to the $\frac{K^2}{R^2}$, where K is the radius of gyration.

$$\therefore h = \frac{v_{CM}^2\left(1 + \frac{k^2}{R^2}\right)}{2\,g}$$

Here, the object is a solid sphere, whose moment of inertia about the centre is:

$$I_{CM} = \frac{2}{5}\,M\,R^2$$

Substituting this value of $\mathbf{I_{CM}}$in equation (1) we get

$$h = \frac{7\,v_{CM}^2}{10\,g}$$

EXAMPLE:

What fraction of the total rolling kinetic energy, of rolling a hollow sphere is due to its rotational motion about its centre of mass?

SOLUTION:

Consider a rolling hollow sphere; its total rolling kinetic energy is given as:

$$KE = \frac{1}{2}\,M\,v_{CM}^2 + \frac{1}{2}\,I_{CM}\,\omega^2 \ldots\ldots\ldots\ldots\ldots \quad . \qquad (1)$$

The rotational part of the kinetic energy, of this sphere, is:

$$\mathbf{KE_{rot}} = \frac{1}{2}\, \mathbf{I_{CM}}\, \omega^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (2)$$

$$\frac{\mathbf{KE_{rot}}}{\mathbf{KE}} = \frac{\frac{1}{2}\, \mathbf{I_{CM}}\, \omega^2}{\frac{1}{2}\, \mathbf{M\, v_{CM}^2} + \frac{1}{2}\, \mathbf{I_{CM}}\, \omega^2} = \frac{\mathbf{I_{CM}}\, \omega^2}{\mathrm{MR^2}\omega^2 + \mathbf{I_{CM}}\, \omega^2}$$

$$\frac{\mathrm{KE_{rot}}}{\mathrm{KE}} = \frac{\frac{I_{CM}}{M\,R^2}}{1+\frac{I_{CM}}{M\,R^2}} = \frac{k^2}{R^2+k^2} \ldots\ldots\ldots\ldots\ldots (3)$$

For a hollow sphere, $k^2 = \frac{I_{CM}}{M} = \frac{\frac{2}{3} M\,R^2}{M} = \frac{2}{3} R^2$

Therefore,

$$\frac{\mathrm{KE_{rot}}}{\mathrm{KE}} = \frac{k^2}{R^2 + k^2} = \frac{2/3}{1 + 2/3} = \frac{2}{5}$$

Equation (3) can be used in general, for all rolling objects. If the fraction of translational kinetic energy is to be determined then:

$$\frac{\mathrm{KE_{trans}}}{\mathrm{KE}} = 1 - \frac{\mathrm{KE_{rot}}}{\mathrm{KE}} = 1 - \frac{k^2}{R^2+k^2} = \frac{R^2}{R^2+k^2}$$

For the hollow sphere, this fraction is:

$$\frac{R^2}{R^2 + \frac{2\,R^2}{3}} = \frac{3}{5}$$

9. SUMMARY

- Rolling motion can be viewed as a motion in which the ‘translation motion’ is superposed on the ‘rotational motion’ of the given object. The net velocity, of the particles on the object will be obtained by adding their two velocity vectors; one due to pure rotation and the other due to pure translation.

i.e. Net velocity of any particle on object

$$\overrightarrow{v_{net}} = \overrightarrow{v_R} + \overrightarrow{v_{CM}}$$

- Here, $\overrightarrow{v_R}$ is the linear velocity of the given particle due to its rotational motion while $\overrightarrow{v_{CM}}$ is the velocity of the particle due to its translational motion.

- A rolling motion, without slipping is called pure rolling. In this case, the translational speed of the centre of mass of the rolling object is related with its angular speed about its centre as
  - $|\overrightarrow{\mathbf{v}_{\mathbf{CM}}}| = \mathbf{R\omega}$
  - Here, $\overrightarrow{v_{CM}}$ is the velocity of translation (i.e.of the centre of mass), R is the radius and ω is the angular speed about the centre of mass of the body.
- During pure rolling, the point of contact will be momentarily at rest i.e. there is no relative motion between the body and the surface in contact. Also the pure rolling object, can be viewed as if it were effectively rotating about the point of contact; this point, therefore acts as the instantaneous centre and the kinetic energy of a (purely) rolling body, is the sum of its kinetic energies of translation and rotation:
  - $KE_{Rolling} = \frac{1}{2} M v_{CM}^2 + \frac{1}{2} I_{CM} \omega^2$
  - Here, the mass of body is M, the velocity of the centre of mass is v_{CM}, and the angular speed is ω.